Velocity-Dependent Eddington Factor in Relativistic Radiative Flow

Jun Fukue

Astronomical Institute, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582

fukue@cc.osaka-kyoiku.ac.jp

Abstract

We propose a variable Eddington factor, depending on the *flow velocity v*, for the relativistic radiative flow, whose velocity becomes of the order of the speed of light. When the gaseous flow is radiatively accelerated up to the relativistic regime, the velocity gradient becomes very large in the direction of the flow. As a result, the radiative diffusion may become *anisotropic* in the comoving frame of the gas. Hence, in a flow that is accelerated from subrelativistic to relativistic regimes, the Eddington factor should be different from 1/3 even in the diffusion limit. As a simple form, the velocity-dependent Eddington factor may be written as $f(\beta) = 1/3 + (2/3)\beta$, where $\beta = v/c$. Using the velocity-dependent Eddington factor, we can solve the rigorous equations of the relativistic radiative flow accelerated up to the relativistic speed. We also propose a generalized form for a variable Eddington factor as a function of the optical depth τ as well as the flow velocity: $f(\tau, \beta) = 1/3 + (2/3)[1 + (\tau + 1)\beta]/(1 + \tau + \beta)$ for a spherically symmetric case. The velocity-dependent Eddington factor can be used in various relativistic radiatively-driven flows, such as black-hole accretion flows, relativistic astrophysical jets and outflows, and relativistic explosions like gamma-ray bursts.

Key words: accretion, accretion disks — astrophysical jets — gamma-ray bursts — radiative transfer — relativity

1. Velocity-Dependent Eddington Factor

In the fundamentals of radiation transfer in the static atmosphere, in order to solve moment equations truncated at the finite order, we need some relation to close the sequence (Chandrasekhar 1960; Mihalas 1970; Rybicki, Lightman 1979; Mihalas, Mihalas 1984, Shu 1991). When moment equations are truncated at the second order, we usually adopt the *Eddington approximation* in the inertial frame (laboratory frame) as a closure relation,

$$P^{ij} = \frac{\delta^{ij}}{3} E, \tag{1}$$

where E and P^{ij} are the radiation energy density and the radiation stress in the inertial frame, respectively. In a relativistic radiative flow, where the gas moves at a relativistic speed, we also assume the Eddington approximation in the comoving frame (fluid frame),

$$P_0^{ij} = \frac{\delta^{ij}}{3} E_0, \tag{2}$$

where E_0 and P_0^{ij} are the radiation energy density and the radiation stress in the comoving frame, respectively. Then, this closure relation in the comoving frame is transformed to the relation in the inertial frame (Lindquist 1966; Hsieh, Spiegel 1976; Fukue et al. 1985; Kato et al. 1998).

These relations are valid in the diffusion limit, where the photon mean-free path is sufficiently smaller than the scalelength of the system in the optically thick regime. In other words, in such a diffusion limit, the radiative

diffusion is supposed to be *isotropic*, and the so-called *Eddington factor f* is set to be

$$f = \frac{1}{3}. \tag{3}$$

However, in the optically thin regime, where the photon mean-free path is sufficiently larger than the scalelength of the system, the radiative diffusion generally becomes *anisotropic*. For example, in the spherically symmetric case, the Eddington factor should be unity in the streaming limit (it should be noted that in the plane-parallel case the Eddington factor is always 1/3). Hence, in order to bridge the optically thick to thin regimes, a variable Eddington factor, which depends on the optical depth τ, is usually adopted,

$$f(\tau) = \frac{1 + \tau}{1 + 3\tau} \tag{4}$$

in the spherically symmetric case (Tamazawa et al. 1975).

Recently, relativistic radiative flows were rigorously solved under a simple situation by Fukue (2005b). He examined a flow radiatively-driven perpendicular to a luminous disk under a fully special relativistic treatment, taking into account radiation transfer. He adopted the current formalism as well as the Eddington approximation in the comoving frame (e.g., Kato et al. 1998). When he arranged the basic equations of relativistic radiation hydrodynamics for the one-dimensional flow without gravity or other forces, he found the *singularity* in the sense of "sonic" points, where the flow speed is equal to the relativistic sound speed of $c/\sqrt{3}$. This singularity origi-

nates from the closure relation (2) for the radiation fields, and the appearance of the singularity suggest that the Eddington approximation would be invalid in the relativistic flow with strong velocity gradients (cf. Turolla, Nobili 1988; Dullemond 1999).

In analogy to the optically thick-thin transition, the possible invalidity of the Eddington approximation in such a relativistic flow can be understood as follows. When the radiative flow is accelerated up to the relativistic regime and the flow velocity becomes of the order of the speed of light, the velocity gradient also becomes very large in the direction of flow. As a result, even in the comoving frame of the gas, the velocity fields as well as density distributions are no longer uniform, and the photon mean-free path would be longer in the downstream direction than in the upstream or other directions. Hence, the radiative diffusion becomes *anisotropic*, and we should consider a variable Eddington factor: the Eddington factor may depend on the *flow velocity* v $(=\beta c)$ in the form such as

$$f(\beta) = \frac{1}{3} + \frac{2}{3}\beta. \tag{5}$$

In several literatures (e.g., Mihalas, Mihalas 1984; Mihalas 1986; Kato et al. 1998), the influence of the flow velocity on radiation transfer was discussed in the context of the description between the comoving and inertial frames. That is, at the lowest order of the flow velocity to the speed of light, the closure relation in the inertial frame must be modified to the order of $(v/c)^1$. In the comoving frame, however, the Eddington approximation (2) was adopted, and no one considered the possible violation of the Eddington approximation in the comoving frame.

In this paper, we thus propose a *velocity-dependent variable Eddington factor* for the relativistic radiative flow, whose velocity becomes of the order of the speed of light.

In the next section we describe the basic equations for the one-dimensional vertical radiative flow, in order to show the typical problem which needs the velocity-dependent Eddington factor. In section 3, we examine several properties required for the velocity-dependent Eddington factor. In section 4, we solve a relativistic radiative flow under the appropriate boundary conditions at the flow base and top, and display the effect of the velocity-dependent Eddington factor. The final section is devoted to the discussion on the physical reason of the present problem, the generalization of the variable Eddington factor, and concluding remarks.

2. Basic Equations

In order to demonstrate the present problem inherent in the relativistic radiation transfer flow, we examine a simple one-dimensional flow in what follows.

We consider the plane-parallel case in the vertical direction; a luminous flat disk or thin skin above a luminous sphere. The radiative energy is transported in the vertical direction, and the gas itself also moves in the vertical direction by the action of radiation pressure. For simplicity, in the present paper, the radiation field is sufficiently intense that both the gravitational field, e.g., of the central object, and the gas pressure are ignored. The internal heating is also ignored. As for the order of the flow velocity v, we consider the fully relativistic regime, where the terms are retained up to the second order of (v/c).

Under these assumptions, the radiation hydrodynamic equations for steady vertical (z) flows are described as follows (Kato et al. 1998; Fukue 2005b).

The continuity equation is

$$\rho c u = J \ (=\text{const.}), \tag{6}$$

where ρ is the proper gas density, u the vertical four velocity, J the mass-loss rate per unit area, and c the speed of light. The four velocity u is related to the proper three velocity v by $u = \gamma v/c$, where γ is the Lorentz factor, $\gamma = \sqrt{1+u^2} = 1/\sqrt{1-(v/c)^2}$.

The equation of motion is

$$c^2 u \frac{du}{dz} = \frac{\kappa_{\rm abs} + \kappa_{\rm sca}}{c} \left[F\gamma(1+2u^2) - c(E+P)\gamma^2 u \right], \tag{7}$$

where $\kappa_{\rm abs}$ and $\kappa_{\rm sca}$ are the absorption and scattering opacities (gray), defined in the comoving frame, E the radiation energy density, F the radiative flux, and P the radiation pressure observed in the inertial frame. The first term in the square bracket on the right-hand side of equation (7) means the radiatively-driven force, which is modified to the order of u^2, whereas the second term is the radiation drag force, which is also modified, but roughly proportional to the velocity.

In the no-gas pressure approximation and without heating, the energy equation is reduced to a radiative equilibrium relation,

$$0 = j - c\kappa_{\rm abs}E\gamma^2 - c\kappa_{\rm abs}Pu^2 + 2\kappa_{\rm abs}F\gamma u, \tag{8}$$

where j is the emissivity defined in the comoving frame. In this equation (8), the third and fourth terms on the right-hand side appear in the relativistic regime.

For radiation fields, the zeroth-moment equation becomes

$$\frac{dF}{dz} = \rho\gamma \left[j - c\kappa_{\rm abs}E + c\kappa_{\rm sca}(E+P)u^2 \right. $$
$$\left. + c\kappa_{\rm abs}Fu/\gamma - \kappa_{\rm sca}F(1+v^2/c^2)\gamma u \right]. \tag{9}$$

The first-moment equation is

$$\frac{dP}{dz} = \frac{\rho\gamma}{c} \left[ju/\gamma - \kappa_{\rm abs}F + c\kappa_{\rm abs}Pu/\gamma \right. $$
$$\left. - \kappa_{\rm sca}F(1+2u^2) + c\kappa_{\rm sca}(E+P)\gamma u \right]. \tag{10}$$

In order to close moment equations for radiation fields, we need some closure relation. If we adopt the usual Eddington approximation,

$$P_0 = \frac{1}{3}E_0 \tag{11}$$

in the comoving frame as the closure relation (Fukue 2005b), where P_0 and E_0 are the quantities in the comoving frame, the transformed closure relation in the inertial frame becomes

$$cP\left(1+\frac{2}{3}u^2\right) = cE\left(\frac{1}{3}-\frac{2}{3}u^2\right) + \frac{4}{3}F\gamma u \qquad (12)$$

(see Kato et al. 1998 for details). Instead, we here adopt a *velocity-dependent* variable Eddington factor $f(\beta)$,

$$P_0 = f(\beta)E_0 \qquad (13)$$

in the comoving frame, where $\beta = v/c$, v being the flow velocity. If we adopt this form (13) as the closure relation in the comoving frame, the transformed closure relation in the inertial frame is

$$cP\left(1+u^2-fu^2\right) = cE\left(f\gamma^2-u^2\right) + 2F\gamma u\left(1-f\right), (14)$$

or equivalently,

$$cP\left(1-f\beta^2\right) = cE\left(f-\beta^2\right) + 2F\beta\left(1-f\right). \qquad (15)$$

The constraint on the function $f(\beta)$ will be discussed later.

Eliminating j with the help of equations (8), and using continuity equation (6), equations (7), (9), and (10) are rearranged as

$$cJ\frac{du}{dz} = (\kappa_{\rm abs}+\kappa_{\rm sca})\rho\frac{\gamma}{c}\left[F(1+2u^2)-c(E+P)\gamma u\right], (16)$$

$$\frac{dF}{dz} = (\kappa_{\rm abs}+\kappa_{\rm sca})\rho u\left[c(E+P)\gamma u-F(1+2u^2)\right], \qquad (17)$$

$$\frac{dP}{dz} = (\kappa_{\rm abs}+\kappa_{\rm sca})\rho\frac{\gamma}{c}\left[c(E+P)\gamma u-F(1+2u^2)\right]. (18)$$

The integration of the sum of equations (16) and (18) yields the momentum flux conservation along the flow,

$$cJu + P = K \ (= {\rm const.}). \qquad (19)$$

Similarly, after some manipulations, the integration of the sum of equations (16) and (17) gives the energy flux conservation along the flow,

$$c^2J\gamma + F = L \ (= {\rm const.}). \qquad (20)$$

At this stage, the basic equations are the equation of motion (16), the mass flux (6), the momentum flux (19), the energy flux (20), and the closure relation (14).

Next, by introducing the optical depth τ by

$$d\tau = -(\kappa_{\rm abs}+\kappa_{\rm sca})\rho dz, \qquad (21)$$

the equation of motion (16) is rewritten as

$$cJ\frac{du}{d\tau} = -\frac{\gamma}{c}\left[F(1+2u^2)-c(E+P)\gamma u\right]. \qquad (22)$$

Furthermore, eliminating E with the help of equation (14), this equation (22) can be finally rewritten as

$$cJ\frac{du}{d\tau} = -\frac{\gamma}{c}\frac{F(f\gamma^2+u^2)-cP(1+f)\gamma u}{f\gamma^2-u^2}, \qquad (23)$$

or equivalently,

$$c^2J\gamma^2\frac{d\beta}{d\tau} = -\frac{F(f+\beta^2)-cP(1+f)\beta}{f-\beta^2}, \qquad (24)$$

where $\beta = v/c$. When the Eddington factor f is $1/3$, equations (23) and (24) are reduced to those given in Fukue (2005b).

We then solve equations (23) [or (24)], (19), and (20) for appropriate boundary conditions at the moving surface, as discussed in Fukue (2005b), and for a suitable form of the variable Eddington factor $f(\beta)$.

3. Properties of $f(\beta)$

We here briefly discuss the constraints on the velocity-dependent variable Eddington factor $f(\beta)$ and choose a preferable candidate for $f(\beta)$.

At first, this function must be reduced to $1/3$ in the non-relativistic limit of $\beta = 0$. In the extremely relativistic limit of $\beta = 1$ with strong velocity gradients, where the photon mean-free path becomes large, this function would approach unity, as in the case of the streaming limit for an optically thin regime. Hence, the function $f(\beta)$ must satisfy the boundary conditions,

$$f(0) = \frac{1}{3}, \qquad (25)$$

$$f(1) = 1, \qquad (26)$$

and should monotonically increase from 0 to 1.

Another constraint comes from the critical conditions. As can be seen in equations (23) and (24), these equations have *singularities* (critical points). That is, when the condition,

$$f(\beta) - \beta^2 = 0, \qquad (27)$$

is satified at some β, the denominator of equation (24) vanishes, and the equation becomes singular. For example, if the Eddington factor is fixed as $f = 1/3$, the equation becomes singular at $\beta = \pm 1/\sqrt{3}$, or at the point, where the flow velocity is equal to the relativistic sound speed of $c/\sqrt{3}$. As already pointed out in Dullemond (1999), this singularity originates from the finite number of moments and the closure relation adopted (see also Fukue 2005b). In order for the relativistic transfer flow to be always regular, the condition,

$$f(\beta) - \beta^2 > 0, \qquad (28)$$

should be satisfied for $0 \le \beta < 1$. Furthermore, since $f = 1$ at $\beta = 1$, the equation is marginally critical ($f - \beta^2 = 0$) at $\beta = 1$.

In addition, from the linear analysis around a critical point of equation (23), the velocity gradient near to the singularity is found to be

$$\left.\frac{du}{d\tau}\right|_{\rm c} = \left.\frac{u+\frac{F\gamma^3}{c^2J}\frac{1-f}{1+f}\frac{\partial f}{\partial u}}{(1-f)2u-(1+u^2)\frac{\partial f}{\partial u}}\right|_{\rm c}, \qquad (29)$$

where the subscript c denotes the values at the critical point. If the Eddington factor is fixed as $f = 1/3$, the velocity gradient becomes $du/d\tau|_{\rm c} = 3/4$ ($d\beta/d\tau|_{\rm c} = 1/\sqrt{3}$). Or, the transonic solution for the outflow ($u > 0$) must be decelerated at the critical point, as was discussed in Fukue (2005b), where the subsonic solutions were obtained. Hence, in order for the flow to be accelerated, even if the solution is marginally critical ($f - \beta^2 = 0$) at $\beta = 1$, the velocity gradient $du/d\tau|_{\rm c}$ should be negative (or zero) there for the outflow ($u > 0$) solutions. Since

the numerator on the right-hand side of equation (29) is always positive, the velocity gradient is negative if the denominator of equation (29) is negative:

$$\frac{\partial}{\partial u}\left[(1-f)(1+u^2)\right]\bigg|_{\rm c} = \frac{\partial}{\partial u}\left[(1-f)\gamma^2\right]\bigg|_{\rm c} < 0, \qquad (30)$$

or equivalently,

$$\frac{\partial}{\partial \beta}\left(\frac{1-f}{1-\beta^2}\right)\bigg|_{\rm c} < 0. \qquad (31)$$

Now, bearing these constraints in mind, we examine several possible candidates of velocity-dependent Eddington factors.

One simple case is a form of

$$f(\beta) = \frac{1}{3} + \frac{2}{3}\beta^n, \qquad (32)$$

where n is constant. In this case, the Eddington factor is a monotonically increasing function of β, which satifies the boundary conditions at $\beta = 0$ and 1. However, since $f - \beta^2$ becomes negative at some β for large values of n ($n \geq 3$), the values of n should be restricted as $n \leq 3$. Moreover, for this form, the equation is marginally critical at $\beta = 1$, and the velocity gradient at the critical point is negative, zero, and positive for $n = 1, 2, 3$, respectively. Hence, the case of $n = 3$ is rejected, and the cases of $n = 1$ and 2 are retained; the simplest case is $n = 1$.

Alternative is a form of

$$f(\beta) = \frac{1}{3 - 2\beta^n}, \qquad (33)$$

where n is constant. In this case, f is also monotonically increasing, but $f - \beta^2$ becomes negative for large n ($n \geq 2$), and the values of n may be around unity. Moreover, for this form, the equation is marginally critical at $\beta = 1$, and the velocity gradient at the critical point is positive, and this form may be inadequate.

In analogy to a variable Eddington factor for the optically thick-thin transition, we can consider a form of

$$f(\beta) = \frac{1 + 1/u}{1 + 3/u} = \frac{\beta + \sqrt{1 - \beta^2}}{\beta + 3\sqrt{1 - \beta^2}}. \qquad (34)$$

In this case, f is monotonically increasing, but $f - \beta^2$ becomes negative at large u. Similar forms such as $f = (1 + 1/u^n)/(1 + 3/u^n)$ or $f = [1 + 1/(\gamma - 1)^n]/[1 + 3/(\gamma - 1)^n]$ would be also inadequate, although we have not checked all possible cases one by one.

Consequently, we adopt the simplest case of

$$f(\beta) = \frac{1}{3} + \frac{2}{3}\beta \qquad (35)$$

in this paper.

4. Examples of Solutions

In this section we show several examples of solutions of equations (24), (19), and (20) with velocity-dependent Eddington factors under appropriate boundary conditions at the moving surface.

As already pointed out in Fukue (2005b), the usual boundary conditions for the static atmosphere cannot be used for the present radiative flow, which moves with velocity at the order of the speed of light.

When there is no motion in the gas ("static photosphere"), the radiation field just above the surface under the plane-parallel approximation is easily obtained. Namely, just above the disk with surface intensity $I_{\rm s}$, the radiation energy density $E_{\rm s}$, the radiative flux $F_{\rm s}$, and the radiation pressure $P_{\rm s}$ are $(2/c)\pi I_{\rm s}$, $\pi I_{\rm s}$, and $(2/3c)\pi I_{\rm s}$, respectively, where the subscript s denotes the values at the disk surface. However, the radiation field just above the surface changes when the gas itself does move upward ("moving photosphere"), since the direciton and intensity of radiation change due to relativistic aberration and Doppler effect (cf. Kato et al. 1998; Fukue 2000).

Let us suppose a situation that a flat infinite photosphere with surface intensity $I_{\rm s}$ in the comoving frame is not static, but moving upward with a speed $v_{\rm s}$ ($= c\beta_{\rm s}$, and the corresponding Lorentz factor is $\gamma_{\rm s}$), where the subscript s denotes the values at the surface. Then, just above the surface, the radiation energy density $E_{\rm s}$, the radiative flux $F_{\rm s}$, and the radiation pressure $P_{\rm s}$ measured in the inertial frame become, respectively,

$$cE_{\rm s} = 2\pi I_{\rm s}\frac{3\gamma_{\rm s}^2 + 3\gamma_{\rm s}u_{\rm s} + u_{\rm s}^2}{3}, \qquad (36)$$

$$F_{\rm s} = 2\pi I_{\rm s}\frac{3\gamma_{\rm s}^2 + 8\gamma_{\rm s}u_{\rm s} + 3u_{\rm s}^2}{6}, \qquad (37)$$

$$cP_{\rm s} = 2\pi I_{\rm s}\frac{\gamma_{\rm s}^2 + 3\gamma_{\rm s}u_{\rm s} + 3u_{\rm s}^2}{3}, \qquad (38)$$

where $u_{\rm s}$ ($= \gamma_{\rm s}v_{\rm s}/c$) is the flow four velocity at the surface (Fukue 2005b).

Thus, we impose the following boundary conditions: At the flow base (deep "inside" the atmosphere) with an arbitrary optical depth τ_0, the flow velocity u is zero, the radiative flux is F_0 (which is a measure of the strength of radiation field), and the radiation pressure is P_0 (which connects with the radiation pressure gradient and relates to the internal structure), where the subscript 0 denotes the values at the flow base. At the flow top ("surface" of the atmosphere) where the optical depth is zero, the radiation field should satisfy the values above a moving photosphere given by equations (36)–(38).
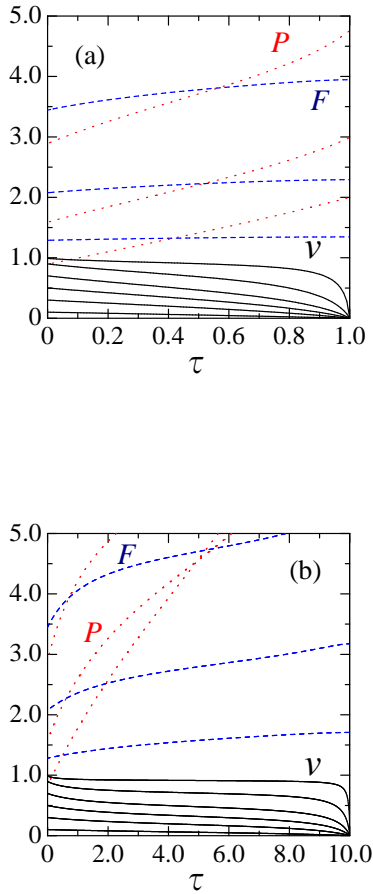
Applying boundary conditions (36)–(38) to equations (19) and (20), we have two relations on the boundary values and mass-loss rate:

$$Jc^2 u_{\rm s} + cP_{\rm s} = cP_0, \qquad (39)$$

$$Jc^2 \gamma_{\rm s} + F_{\rm s} = Jc^2 + F_0. \qquad (40)$$

Physically speaking, in the radiative flow starting from the flow base with an arbitrary optical depth τ_0, for initial values of F_0 and P_0 at the flow base, the final values of the radiation fields $E_{\rm s}$, $F_{\rm s}$, $P_{\rm s}$, and the flow velocity $v_{\rm s}$ at the flow top can be obtained by solving basic equations. Furthermore, the mass-loss rate J is determined as an eigenvalue so as to satisfy the bondary condition at the flow top (cf. Fukue 2005a in the subrelativistic regime).
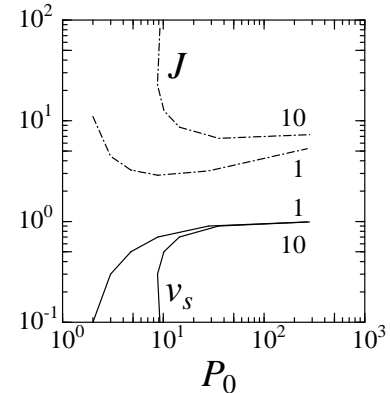
Fig. 2. Final velocity v_s at the flow top (solid curves), and the mass-loss rate J (chain-dotted ones), as a function of P_0 for several values of τ_0 at the flow base: $\tau_0 = 1$ and 10. The quantities are normalized in units of c and πI_s. That is, the unit of F and cP is πI_s and the unit of J is $\pi I_s/c^2$.

Fig. 1. Flow three velocity v (solid curves), radiative flux F (dashed curves), and radiation pressure P (doted curves), as a function of the optical depth τ for several values of v_s at the flow top in a few cases of τ_0: (a) $\tau_0 = 1$ and (b) $\tau_0 = 10$. The values of v_s are 0.1, 0.3, 0.5, 0.7, 0.9, and 0.99 from bottom to top of v and 0.1, 0.3, and 0.5 from bottom to top of F and P. The velocity v is in units of c, F and cP in πI_s.

In the present, fully relativistic case, however, the final values of the radiation fields at the flow top depend on the flow velocity there, and the final values at the flow top cannot be analytically expressed by the initial values at the flow base. Hence, in this paper we determine the mass-loss rate as follows.

In the radiative flow with optical depth τ_0, we first give the final flow velocity v_s (and γ_s), instead of the initial value of P_0. Then, the final values of radiation fields E_s, F_s, and P_s can be fixed by equations (36)–(38). Next, we give a trial value for the mass-loss rate J, and the initial values of P_0 and F_0 can be fixed by equations (39) and (40). Since all the parameters are temporarily fixed, we solve equation (24) from $\tau = \tau_0$ to $\tau = 0$. Generally, however, the obtained final velocity at $\tau = 0$ is different from a given v_s. Thus, we vary the value of J and follow iterative processes, so that the calculated final velocity

coincides with a given final velocity v_s.

Finally, we adopt the velocity-dependent Eddington factor of the form (35).

Examples of the relativistic radiative flows under the present velocity-dependent Eddington factor are shown in figures 1 and 2.

In figure 1 we show the flow three velocity v (solid curves), the radiative flux F (dashed curves), and the radiation pressure P (doted curves), as a function of the optical depth τ for several values of v_s at the flow top and for $\tau_0 = 1$ and 10. The quantities are normalized in units of c and πI_s.

In contrast to Fukue (2005b), where the acceleration solution beyond $c/\sqrt{3}$ was not obtained by the presence of the singularity, originating from the usual fixed Eddington factor, the present flow velocity can easily exceed $c/\sqrt{3}$ to be accelerated toward the highly relativistic regime, since the singularity is evaded by the adoption of the velocity-dependent Eddington factor (35).

Other properties of the flow are similar to the previous case. For example, when the initial radiative flux F_0 at the flow base is large, the flow is effectively accelerated, and the final speed of the flow becomes large. The mass-loss rate J is not given arbitrarily, but is determined as an eigenvalue (Fukue 2005a, b).

In figure 2 we show the final velocity v_s at the flow top (solid curves) and the mass-loss rate J (chain-dotted ones), as a function of P_0 for several values of τ_0 at the flow base. The quantities are normalized in units of c and πI_s. For example, the unit of J is $\pi I_s/c^2$.

As can be seen in figure 1, as the radiative flux increases, the final flow velocity at the flow top increases. Moreover, as can be seen in figure 2, and similar to the previous case (Fukue 2005a, b), in order for the flow to exist, the radiation pressure P_0 at the flow base is restricted in some range. In the subrelativistic case without gravity and
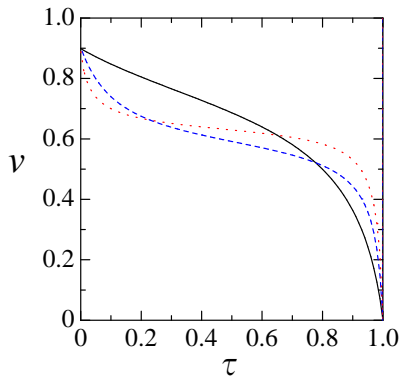
Fig. 3. Flow three velocity v in several forms of velocity-dependent Eddington factors. The parameters are $\tau_0 = 1$ and $v_{\rm s} = 0.9\ c$.

heating (Fukue 2005a), the initial pressure P_0 is proved to be restricted in the range of $2/3 < cP_0/F_{\rm s} < 2/3 + \tau_0$. In the present case, the initial pressure is also restricted in some range, but is modified due to the relativistic effect and the velocity-dependent Eddington factor. In addition, the loaded mass increases as the initial optical depth increases, even if the final flow speeds are the same.

For comparison, we show in figure 3 the results for several other forms of velocity-dependent Eddington factors. In figure 3, the parameters are fixed as $\tau_0 = 1$ and $v_{\rm s} = 0.9\ c$. A solid curve is for the case of $f(\beta) = 1/3 + 2\beta/3$, whereas a dashed one is for the case of $f(\beta) = 1/3 + 2\beta^2/3$, and a dotted one is for the case of $f(\beta) = 1/(3 - 2\beta)$. The boundary conditions at the flow top are the same in each case, but the radiation fields (F_0 and P_0) at the flow base and the loaded mass J are different in each case. In particular, the loaded mass in the case of $f(\beta) = 1/3 + 2\beta/3$ is larger than those in other cases examined, and this is another reason to prefer the present form.

It should be finally noted that, thanks to the velocity-dependent Eddington factor, the relativistic radiative flow can be accelerated to reach an extremely relativistic regime, and the barrier of the magic speed above the static atmosphere (Icke 1989) would be now completely cleared away.

5. Discussion and Concluding Remarks

Moment equations for relativistic radiation transfer have been derived in several literature (Lindquist 1966; Anderson, Spiegel 1972; Hsieh, Spiegel 1976; Thorne 1981; Udey, Israel 1982; Schweizer 1982). A complete set of moment equations for relativistic flow is given by the projected, symmetric, trace-free formalism by Thorne (1981). Since a moment expansion gives an infinite set of equations, one must truncate the expansion at the finite order by adopting a suitable closure assumption, in order to make the transfer problem tractable. When one truncate

at the second order, for example, the Eddington approximation is usually adopted as a closure relation.

Such radiation moment formalism is quite convenient, and it is a powerful tool for tackling problems of relativistic radiation hydrodynamics. However, its validity is never known unless fully angle-dependent radiation transfer equation is solved.

Actually, the pathological behavior in relativistic radiation moment equations has been pointed out (Turolla, Nobili 1988; Dullemond 1999). Namely, moment equations for relativistic radiation transfer can have singular (critical) points. For example, in the one-dimensional relativistic radiation flow using the Eddington approximation (2), where the moment equations are truncated at the second order, the singularity appears when the flow velocity becomes $\beta = v/c = 1/\sqrt{3}$ (e.g., Fukue 2005b). As a result, solutions behave pathologically in regions of strong velocity gradients. The appearance of singularities is explained because we approximate the full transfer equations with a finite number of moments (Dullemond 1999).

In order to avoid the singularity, in this paper, we thus propose and examine a *velocity-dependent variable Eddington factor* in radiation transfer in relativistic radiative flows, which are accelerated up to a relativistic speed with strong velocity gradients.

As already stated in section 1, the physical reason is clear. In an optically thick, low velocity regime, the mean free path of photons are the same in all directions, and the radiative diffusion is isotropic. In a relativistically accelerated flow, however, the velocity gradient of the flow becomes very large, and the density distribution also becomes non-uniform even in the comoving frame of the gas. Hence, the mean free path becomes longer in the downstream direction than in the upstream and other directions, even in an optically thick regime, and the radiative diffusion becomes anisotropic. Thus, in the case of sub-relativistic to relativistic regimes, as in the case of optically thick to thin regimes, we should consider a variable Eddington factor which depends on the flow velocity v. The most preferable one, we propose, is

$$f(\beta) = \frac{1}{3} + \frac{2}{3}\beta, \tag{41}$$

where $\beta = v/c$.

By adopting the velocity-dependent Eddington factor, the singularity found under the traditional formalism is removed, and the relativistic radiative flow can be solved from a low velocity regime to an extremely relativistic regime. It should be noted, however, that the present approach is relevant only for moment expansion up to the second order. If we attempt to solve the moment equation up to higher order, we may search alternative ways.

In this paper, we considered only the one-dimensinal case. The vector form of the present variable Eddington factor may be written as

$$\boldsymbol{f}(\boldsymbol{\beta}) = \left(\frac{1}{3} - \frac{1}{3}\beta, \frac{1}{3} - \frac{1}{3}\beta, \frac{1}{3} + \frac{2}{3}\beta \right), \tag{42}$$

when the velocity components are $(0,0,\beta)$.

We also propose a general form of a variable Eddington factor $f(\tau, \beta)$, which would be useful for an arbitrary optical depth τ, and for an arbitrary velocity $v (= \beta c)$. The conditions for such a generalized Eddington factor are: (i) in the low velocity limit ($\beta \sim 0$), it has the similar form as equation (4), (ii) in the optically thick limit ($\tau \gg 1$), it becomes the presently proposed form, (iii) in the optically thick, low velocity limit, it is $1/3$, and (iv) in the optically thin limit ($\tau \sim 0$) and/or in the relativistic limit ($\beta \sim 1$), it approaches unity. For a spherically symmetric case, in order to satisfy these conditions, we apply the relativistic summation rule for the flow speed v and the photon diffusion speed $c/(\tau+1)$, and obtain a generalized Eddington factor as

$$ f(\tau, \beta) = \frac{1}{3} + \frac{2}{3} \frac{\frac{1}{\tau+1} + \beta}{1 + \frac{\beta}{\tau+1}} = \frac{1}{3} + \frac{2}{3} \frac{1 + (\tau+1)\beta}{1 + \tau + \beta}. \quad (43) $$

It should be noted that, in the low velocity limit ($\beta \sim 0$), this hybrid form (43) becomes $f(\tau, 0) = (3 + \tau)/(3 + 3\tau)$, which is somewhat different with equation (4), but essentially the same. In the optically thick limit ($\tau \gg 1$), this equation (43) is reduced to equation (35). For a plane-parallel case, on the other hand, equation (41) can be used for an arbitrary optical depth.

In the present paper, we examine the plane-parallel case with the form (41). The spherically symmetric case using the hybrid form (43) will be discussed in a seperate paper.

The velocity-dependent variable Eddington factor for relativistic radiative flows proposed in the present paper is fundamentally important in various aspects of relativistic astrophysics with radiation transfer. For example, the present form may be applied to the cases of black-hole accretion flows with supercritical accretion rates, relativistic jets and winds driven by luminous central objects, relativistic explosions including gamma-ray bursts, neutrino transfers in supernova explosions, and various events occured in the proto universe.

Of course, the present proposition is only the first one, and not the final one. There exist many points to be improved, including more suitable forms for velocity-dependent Eddington factors. For example, the Eddington factor may depend not on the velocity, but on the velocity gradient. In this field of relativistic radiation tranfer, there still remain many problems to be solved.

References

Anderson, J.L., Spiegel, E.A. 1972, ApJ, 171, 127
Chandrasekhar, S. 1960, Radiative Transfer (New York: Dover Publishing, Inc.)
Dullemond, C.P. 1999, A&A, 343, 1030
Fukue, J. 2000, PASJ, 52, 613
Fukue, J. 2005a, PASJ, 57, 841
Fukue, J. 2005b, PASJ, in press
Fukue, J., Kato, S., & Matsumoto, R. 1985, PASJ, 37, 383
Hsieh, S.-H., Spiegel, E.A. 1976, ApJ, 207, 244
Icke, V. 1989, A&A, 216, 294
Kato, S., Fukue, J., & Mineshige, S. 1998, Black-Hole Accretion Disks (Kyoto: Kyoto University Press)
Lindquist, R.W. 1966, Ann. Phys., 37, 487
Mihalas, D. 1970, Stellar Atmospheres (San Francisco: W.H. Freeman and Co.)
Mihalas, D. 1986, in Astrophysical Radiation Hydrodynamics, ed. K.-H. A. Winkler & M.L. Norman (D. Reidel Publishing Co.)
Mihalas, D., & Mihalas, B.W. 1984, Foundations of Radiation Hydrodynamics (Oxford: Oxford University Press)
Rybicki, G.B., & Lightman, A.P. 1979, Radiative Processes in Astrophysics (New York: John Wiley & Sons)
Schweizer, M.A. 1982, ApJ, 258, 798
Shu, F.H. 1991, The Physics of Astrophysics Vol. 1: Radiation (California: University Science Books)
Tamazawa, S., Toyama, K., Kaneko, N., & Ôno, Y. 1975, ApSpSci, 32, 403
Thorne, K.S. 1981, MNRAS, 194, 439
Turolla, R., Nobili, L. 1988, MNRAS, 235, 1273
Udey, N., & Israel, W. 1982, MNRAS, 235, 1273